Bi-parameter CGM model for approximation of α-stable PDF

X.T. Li, J. Sun, L.W. Jin and M. Liu

Alpha stable distribution has no closed-form expression for the probability density function. Presented is a very concise approximate model for symmetric α-stable ($S\alpha S$) distribution, which is basically a simplified version of the Cauchy-Gaussian mixture (CGM). The proposed model enjoys the advantages of good fitness and analytical tractability.

Introduction: The non-Gaussian noise in the practical world can be well characterised by its impulsive nature. Typical impulsive interference may include atmospheric noise, ambient acoustic noise and lightning, switching transients, etc. Since 1993, there has been tremendous interest in the class of symmetric α-stable ($S\alpha S$) distributions [1], as a generalisation of Gaussian distribution, which can model a wide range of phenomena of varying degrees of impulsivity [2]. In a detection and denoising region, optimal processing is feasible if the noise probability density function (PDF) is analytically known and tractable. Unfortunately, such an attractive model for impulsive noise has no closed forms for the probability densities except for two special cases [1]: Gaussian and Cauchy distribution. This has limited further applications of the α-stable distribution family.

Previous work: Numerically the α-stable PDF can be obtained by taking the inverse Fourier transform of characteristic function [1]; however, it does not provide an analytic form, thus is not suitable for real-time applications owing to the intensive computation. Although asymptotic series are available for $S\alpha S$ density function with $\alpha > 1$, these asymptotic series expansions [1] fit well only in the tails and the neighbourhood of the origin of the PDF, while for intermediate values the approximation deviates from the actual PDF values [3]. Moreover, they are not convenient for processing of the tremendous series. Comparatively, the approximate mixture model is a more feasible method. Currently, there are two classes of approximation mixture models, one is a scale mixture of the Gaussian PDF, which approximates the PDF by a finite number of the Gaussian mixture model (GMM) [2, 3]. GMM fits the $S\alpha S$ distribution well but it cannot capture the algebraic tails of alpha stable distributions with small numbers of Gaussian components N. To achieve accurate approximation, N usually takes a value larger than 8 [2], hence loses the analytical convenience. The other method is a Cauchy Gaussian mixture model (CGM), the PDF of which is given by [4]

$$f(x) = (1-\varepsilon)f_G(x) + \varepsilon f_C(x) = (1-\varepsilon)\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left(-\frac{x^2}{2\sigma_g^2}\right) + \frac{\varepsilon\gamma}{\pi(x^2+\gamma^2)} \quad (1)$$

where ε is the mixture ratio, σ_g^2 is the Gaussian variance and γ is the Cauchy parameter. Using the EM algorithm, Swami achieved the parameters for estimation of such a model [4]. However, the complexity of Swami's approach is somewhat high owing to the iterative estimation for the triple parameter (ε, σ_g, γ). In order to develop more tractable approximation models, we introduce a simplified CGM with only two parameters, which fits $S\alpha S$ density well, and meanwhile is of very low complexity and more analytical convenience.

Bi-parameter CGM model: To achieve an approximation model with less computational burden, we considered a simplified Cauchy Gaussian mixture model with a bi-parameter (ε, σ), which is defined as

$$f(x) = (1-\varepsilon)f_G(x) + \varepsilon f_C(x) = (1-\varepsilon)\frac{1}{2\sqrt{\pi}\sigma}\exp\left(-\frac{x^2}{4\sigma^2}\right) + \frac{\varepsilon\sigma}{\pi(x^2+\sigma^2)} \quad (2)$$

where ε is the mixture ratio, and σ is the scale exponent of the $S\alpha S$ distribution. We call such a CGM model as bi-parameter Cauchy-Gaussian mixture (BCGM) models. To obtain the appropriate value of the ratio parameter ε, McCulloch proposed an empirical equation $\varepsilon = 2 - \alpha$ for $\sigma = 1$ [5]. However, such a linear relation between ε and α is not optimum in a maximum likelihood (ML) sense. Here, we propose another equation with more accuracy by utilising the fractional lower order moment (FLOM). Let X be a real $S\alpha S$ random variable, then

$$E(|x|^p) = \int_{-\infty}^{\infty} |x|^p[(1-\varepsilon)f_G(x) + \varepsilon f_C(x)]dx \quad (3)$$

where $p < \alpha$. For Cauchy and Gaussian distribution, define $m_C^p = \int_{-\infty}^{\infty} |x|^p f_C(x)dx = C(p,1)\sigma^p$ and $m_G^p = \int_{-\infty}^{\infty} |x|^p f_G(x)dx = C(p,2)\sigma^p$, respectively. Thus (3) can be rewritten as $E(|x|^p) = (1-\varepsilon)\, m_G^p + \varepsilon m_C^p$. Then it is easy to see that $\varepsilon = (E(|x|^p) - m_G^p)/(m_C^p - m_G^p)$. Utilising the equality $E(|x|^p) = C(p,\alpha)\sigma^p$ where

$$C(p,\alpha) = \frac{2^{p+1}\Gamma(p+1)/2)\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)}$$

[1], we further have

$$\varepsilon = \frac{C(p,\alpha)\sigma^p - C(p,2)\sigma^p}{C(p,1)\sigma^p - C(p,2)\sigma^p} = \frac{2^{p+1}\Gamma(p+1/2)\Gamma(-p/\alpha)/\left[\alpha\sqrt{\pi}\Gamma(-p/2)\right] - 2^{p+1}\Gamma(p+1/2)/2\sqrt{\pi}}{2^{p+1}\Gamma(p+1/2)\Gamma(-p)/\left[\sqrt{\pi}\Gamma(-p/2)\right] - 2^{p+1}\Gamma(p+1/2)/2\sqrt{\pi}} \quad (4)$$

After straight manipulations we can get $\varepsilon = [2\Gamma(-p/\alpha) - \alpha\Gamma(-p/2)]/[2\alpha\cdot\Gamma(-p) - \alpha\Gamma(-p/2)]$. To test the effectiveness of the above equation, we adopt the ML estimate value as the benchmark, and compare the result of our proposed method with the empirical equation by McCulloch. Fig. 1 shows that our method utilising FLOM matches the ML results very well and is better than the results by the McCulloch equation.

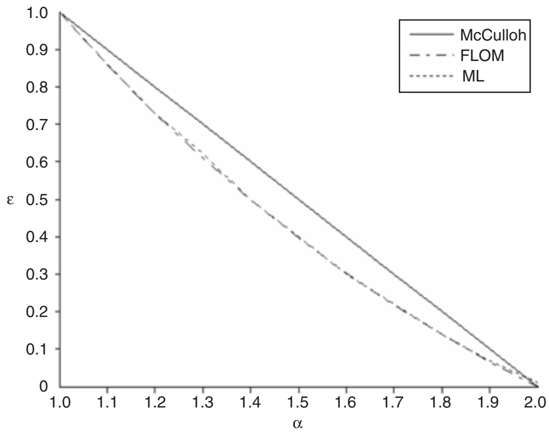

Fig. 1 *Mixture ratio ε against α value ranging from $\alpha = 1$ to $\alpha = 2$*

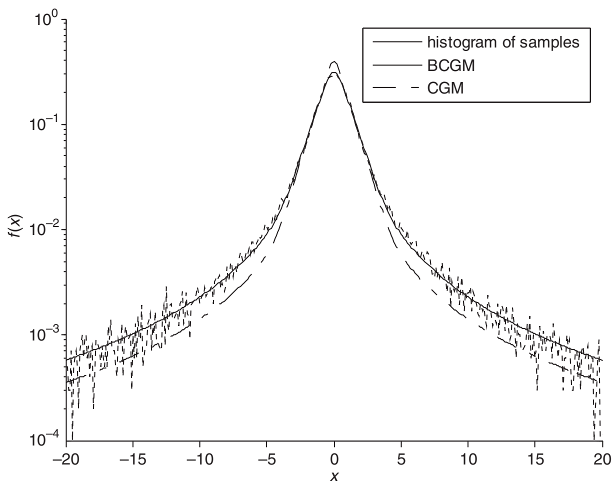

Fig. 2 *Comparison between CGM (KL = 0.2172) and BCGM (KL = 0.1120) in case of $\alpha = 1.2$*

Performance evaluation: To compare the performances of various approximations, KL distance [2] is introduced as the measure. First, we generate IID standard $S\alpha S$ sample series of length 10^5 by the generator proposed by Chambers *et al*. in [6] and evaluate the histogram of samples. Next, we compare the approximate PDF's with the empirical PDF (i.e. histogram) and calculate respective KL distances. In the case of $\alpha = 1.2$, which is close to the Cauchy distribution, comparison between CGM and BCGM is as shown in Fig. 2. The result illustrates that the proposed BCGM model (KL = 0.1120) is superior to the CGM (KL = 0.2172). In another case of $\alpha = 1.8$, which is relatively close to the Gaussian distribution, the approximation performances of BCGM and GMM are as shown in Fig. 3. The KL distance of the BCGM model is slightly greater than that of GMM with $N = 10$, which are 0.0715 and 0.0186, respectively. However, BCGM has a more tractable form than GMM has in that the parameters of the proposed BCGM have closed-form expression.

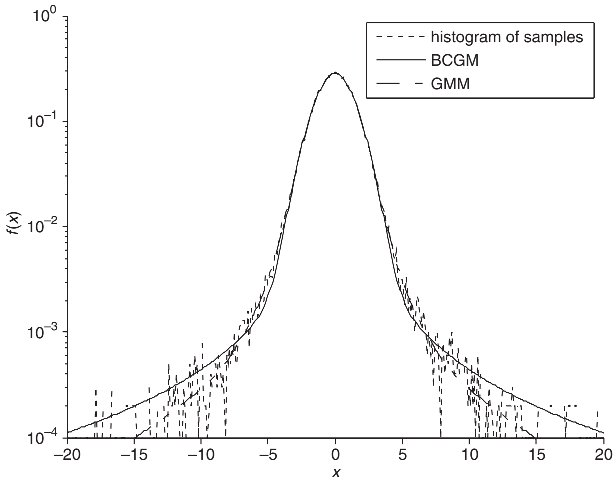

Fig. 3 *Comparison between GMM (KL = 0.0186) and BCGM (KL = 0.0715) in case of* $\alpha = 1.8$

Conclusion: We propose a novel bi-parameter Cauchy-Gaussian mixture model which can well approximate the $S\alpha S$ probability density. The main feature of the BCGM model lies in the very concise expression, which makes it better for tasks such as signal detection and denoising if compared with other popular approximation models such as GMM.

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